Simple Antiflexible Rings

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Abstract— Let R be an antiflexible ring of characteristic ≠ 2,3. In this paper, first we prove that associator is in the center. Using this, we prove that a simple not associative and not commutative antiflexible ring R is right alternative

Index Terms— Associator, antiflexible ring, characteristic, commutator, Lie ring, right alternative

1. INTRODUCTION

Antiflexible albebras were introduced by Kosier [1] , and a subclass of antiflexible rings was studied earlier by Kleinfeld [2]. Thedy [3] proved that simple rings satisfying (a,(b,c,d)) = 0 are either associative or commutative. In [4], K. Survana established the result (R,(R,R,R)) = 0 in (-1,1) ring of characteristic ≠ 2,3. In [5], K. Subhashini proved that a simple not associative and not commutative (1,0) ring R of characteristic ≠ 2,3 is alternative by using (R,(R,R, R)) = 0. In this paper, we prove that a simple not associative and not commutative antiflexible ring R is right alternative.

2. PRELIMINARIES

The associator (x, y, z) and commutator (x, y) are defined by (x, y, z) = (x y) z – x (y z)and (x, y) = x y – y x for all x, y, z in R respectively. The nucleus N of a ring is defined as N = { n ϵ R / (n, R, R) = (R, n, R) = (R, R, n) = 0 }. The center C of R is defined as C = {c ϵ N / (c, R) =0}. A ring is called simple if $R^2 \neq 0$ and the only nonzero ideal of R is itself. A right alternative ring R is a ring in which y (x x) = (y x) x, for all x,y in R.

The ring R is said to be Antiflexible if A(x, y, z) = (x, y, z) – (z, y, x) = 0 ----------------------(1)
is an identity in R.

Throughout this paper we assume that R is antiflexible.

A ring R is of characteristic ≠n if nx =0 implies x = 0 for all x ϵ R and n is a natural number when n = 2, 3 and that the third power associativity condition (x, x, x) = 0 --------------------------(2)
is an identity in R.

With aid of (1), we obtain the identity as a linearization of (2)

B(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0 ------------------------(3)

We shall also require the Teichmuller identity (which holds in any ring)

C(w, x, y, z) = (wx, y, z) – (w, xy, z) + (w, x, yz) – w(x, y, z) – (w, x, y)z = 0 ------------------------(4)

In any ring, (xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) – (x, z, y) holds.

From which we subtract A (x, z, y) + B(x, y, z) = 0

to obtain D(x, y, z) = (xy, z) – x(y, z) – (x, z)y + 2(x, z, y) = 0 -------------------------(5)

We let x o y = x y + y x;

then it can be verified that in any ring

(x o y) o z – x o (y o z) = (x, y, z) + (x, z, y)+ (y, x, z) - (y, z, x) – (z, x, y) – (z, y, x) + (y, (x, y))

so that from (1) we get (x o y) o z – x o (y o z) = (y, (x, z)) -------------------(6)

If we retain the additive group of R but replace the product xy of R by the product xoy, then we obtain a commutative ring R^+, and it follows from (2) and (6) that R^+ is a Jordan Ring. Equally, we could get an anticommutative ring R^- by replacing the product xy by the commutator product (x, y) and from

0 = D(x, y, z) – D(y, x, z) and 0 = A(x, z, y).

It would follow that ((x, y), z) + ((y, z), x) + ((z, x), y) = 0 -----------------------(7)

so that R^- would be a Lie ring.

Expanding 0 = C (w, x, y, z) – C (z, y, x, w) and using

0 = A(z, y, xw) = A (z, yx, w) = A(zy, x, w) = z A(y, x, w) = A(z, y, x)w, we get

0 = E(w, x, y, z)=((w, x), y, z) – (w, (x, y), z) + (w, x, (y, z)) – (w, (x, y, z)) – ((w, x, y), z) -----------------------(8)

Then we expand

0 = E(w, x, y, z) + E(x, y, z, w) + E(y, z, w, x) + E(z, w, x, y) – B((w, x), y, z) – B((x, y), z, w) – B((y ,z), w, x) – B((z, w), x, y)

To get

0 = F (w, x, y, z) = (w, (x, y), z) + (x, (y, z), w) + (y, (z, w), x) + (z, (w, x), y) ---------------------------(9)

Now we are able to derive the important identity $(w, (x, y), z) = 0$ --------------------------(10)

Expanding

$0 = E(x, x, y, x) + E(y, x, x, x) - B(x, x, (y, x)) + (B(x, x, y), x),$

We get, $0 = (x, (x, y, x)),$

Hence from $0 = (x, B(x, y, x))$ and $0 = (x, A(x, x, y))$, we have

$0 = (x, (x, y, x)) = (x, (x, x, y)) = (x, (y, x, x))$ ----------------------------(11)

From (3) and (4), $G(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y,(z, w, x)) - (y,(w, x, y)) = 0$ ----------------------------(12)

$G(x, y, x, y) = (x, (y, x, y)) - (y,(x, y, x)) + (x, (y, x, y) - (y,(x, y, x)) = 0$

$\Rightarrow$ $2 (x,(y, x, y)) - 2 (y,(x, y, x) = 0$

$\Rightarrow$ $(x, (y, x, y)) - (y, (x, y, x)) = 0$

From (1), $A(x, y, y) = 0$ & $A(y, x, x) = 0$ &

Thus $(x, (y, x, y)) - (y, (y, x, x)) = 0$

Combining this with $G(x, y, y, x) = 0$

Gives $2 (x, (x, y, y)) = 0$

And therefore $(x,(x, y, y)) = 0$ ----------------------(13)

Linearize (13) $J(x, y, z) = (x, (x, y, z)) + (x, (x, z, y)) = 0$ ---------------------(14)

Again linearize (13)

$\Rightarrow K(x, y, z) = (x, (z, y, y)) + (z, (x, y, y)) = 0$ ----------------------(15)

Combination of

$K(y, x + y, z) - (y, B(y, x, z)) - K(y, x, z) - (y, A(x, y, z)) - K(z, y, y) + (y, A(z, y, x)) - J(y, z, x) - (z, A (y, x, y)) + K (x, y, z) = 0$ gives

$(y, (z, x + y, x + y)) + (z, (y, x + y, x + y)) - (y, (x, y, z) + (y, z, x) + (z, x, y)) - (y,(z, x, x)) - (z, (y, x, x)) - (y, (x, y, z) - (z, y, x)) -$
$(z, (y, y, y)) - (y, (z, y, y)) + (y, (z, y, x) -(x, y, z)) - (y, (y, z, x)) - (y, (y, x, z)) - (z, (y, x, y) - (y, x, y)) + (x, (z, y, y)) + (z, (x, y, y)) = 0.$

$\Rightarrow (y, (z, x, x)) + (y, (z, x, y)) + (y, (z, y, x)) + (y, (z, y, y)) + (z, (y, x, x)) + (z, (y, x, y)) + (z, (y, y, x)) + (z, (y, y, y)) - (y, (x, y, z)) -$
$(y, (y, z, x)) - (y, (z, x, y)) - (y, (z, x, x) - (z, (y, x, x)) - (y, (x, y, z)) + (y, (z, y, x)) - (z, (y, y, y)) - (y, (z, y, y)) + (y, (z, y, x)$
$- (y, (x, y, z)) - (y, (y, z, x)) - (y, (y, x, z)) + (x, (z, y, y)) + (z, (x, y, y)) = 0$

$\Rightarrow$ $2(z, (y, y, x)) + (z, (y, x, y)) + (x, (z, y, y)) - 2(y, (y, z, x)) - (y, (z, x, y)) = 0$

$\Rightarrow$ $(z, (y, x, y)) + (z, (y, y, x)) - 2 (y, (y, z, x)) - (y, (z, x, y)) = 0$

$\Rightarrow$ $-(z, (x, y, y)) - 2 (y, (y, z, x)) - (y, (z, x, y)) = 0$

$\Rightarrow$ Therefore $L(x,y,z) = -(z,(x,y,y)) - (y,(y,z,x)) + (y,(x,y,z)) = 0$ - ---------------------(16)

Using (1) and (12) can be written as $(x, (y, y, x)) = 0$ ---------------------(17)

Linearization of this equation gives

$(x, (z, y, x)) + (x, (y, z, x)) = 0$

$\Rightarrow$ $(x, (z, y, x)) = - (x, (y ,z, x))$ --------------------(18)

Linearize equation (13)

$(x, (x, y, z)) + (x, (x, z, y)) = 0$

$\Rightarrow (x, (x, y, z)) = - (x, (x, z, y))$ --------------(19)

$(x, (x, y, z)) = - (x, (y, x, z))$ (by (18))

$= - (x, (z, x, y))$ (by (1))

$= (x, (z, y, x))$ (by (18)) -----------------(20)

Commute equation (3) with x

$(x, (x, y, z)) + (x, (y, z, x)) + (x, (z, x, y)) = 0$

$\Rightarrow$ $(x, (z, x, y)) = 0$ (By (20) & (18)) -----------------(21)

From (16) and (19), we have

$(z, (x, y, y)) = 0$

$\Rightarrow$ $(R, (x, y, y)) = 0$ -------------(22)

Linearize (22), we have

$(w, (x, y, z)) + (w, (x, z, y)) = 0$

$\Rightarrow (w, (x, y, z)) = - (w, (x, z, y))$ ------------------(23)

$(w, (x, y, z)) = (w, (z, y, x))$ (by (1))

$= - (w, (z, x, y))$ (by (23))

$= - (w, (y, x, z))$ (by (1))

$= (w, (y, z, x))$ (by (23))

Commute equ(3) with w

(w, (x, y, z)) + (w, (y, z, x)) + (w, (z, x, y)) = 0

$\Rightarrow$ (w, (x,y,z)) = 0 -----------------(24)

3. Main Section:

Let us define T ={ t ϵ R / (t, R) = 0 = (t, R, R) = 0}

Lemma 3.1 : T is an ideal of R.

Proof: To prove T is an ideal.

Substitute x = t in (24) that gives

((t,y,z),w) = 0

From this, we have

(ty.z,w) - (t.yz,w) =0

And this becomes

(ty.z,w) = 0 from the definition of T

Thus ty ϵ T and so T is a right ideal.

However yt ϵ T.

Thus T is a two sided ideal of R.

Theorem3.2 : A simple not associative and not commutative antiflexible ring R of characteristic ≠ 2,3 is right alternative.

Proof: First we prove the identity (r, (z, y, y)w) = 0.

Commute Teichmuller identity C (w, x, y, z) = 0 with r and applying equation (24), we get

-(r, w(x, y, z)) – (r, (w, x, y)z) = 0

Gives (r, w(x, y, z)) = - (r, (w, x, y)z)

If we put y = z = x in this equation, then from (1) it reduces

(r, w(x, x, x)) = - (r, (w, x, x)x)

= 0

$\Rightarrow$ (r, (w, x, x)x) = 0 --------------(25)

From (24) & (25) all (w,x,x) are in T

Since R is simple and T is an ideal of R,

Then either T = R or T = 0.

If T = R then R is commutative.

Since R is not commutative we must have T = 0.

Then (w, x, x) = 0 ------------------(26)

This implies R is Right–alternative.

References :

1. Kosier, Frank, On a class of nonflexible algebras, Trans. Am. Math. Soc. 102 (1962) 298-318.
2. Klenfeld. E, Kosier. F, Osborn, J.M., and Rodabaugh.D, The structure of associator dependent rings, Trans. Am. Math.Soc. 110 (1964), 473-483.
3. A. Thedy, On Rings satisfying ((a,b,c),d) = 0 , Proc.Amer.Math. Soc. 29 (1971), 213 – 218.
4. K. Survana and K. Subhashini, A Result on prime (-1,1) rings Actaciencia Indica, VolXXVI M, no.2 (2000), 85-86.
5. Subhashini .K, Simple (1,0) Rings, International Mathematical Forum, Vol 7,2012, no.48, 2407 -2410.

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